VORTEX PERTURBATIONS OF THE NONSTATIONARY MOTION OF A LIQUID WITH A FREE BOUNDARY

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The first studies on the stability of nonstationary motions of a liquid with a free boundary were published relatively recently [1-4]. Investigations were conducted concerning the stability of flow in a spherical cavity [1, 2], a spherical shell [3], a strip, and an annulus of an ideal liquid. In these studies both the fundamental motion and the perturbed motion were assumed to be potential flow. Changing to Lagrangian coordinates considerably simplified the solution of the problem. Ovsyannikov [5], using Lagrangian coordinates, obtained equations for small potential perturbations of an arbitrary potential flow. The resulting equations were used for solving typical examples which showed the degree of difficulty involved in the investigation of the stability of nonstationary motions [5-8]. In all of these studies the stability was characterized by the deviation of the free boundary from its unperturbed state, i.e., by the normal component of the perturbation vector. In the present study we obtain general equations for small perturbations of the nonstationary flow of a liquid with a free boundary in Lagrangian coordinates. We find a simple expression for the normal component of the perturbation vector. In the case of potential mass forces the resulting system reduces to a single equation for some scalar function with an evolutionary condition on the free boundary. We prove an existence and uniqueness theorem for the solution, and, in particular, we answer the question of whether the linear problem concerning small potential perturbations which was formulated in [5] is correct. We investigate two examples for stability: a) the stretching of a strip and b) the compression of a circular cylinder with the condition that the initial perturbation is not of potential type.

1. FORMULATION OF THE PROBLEM WITH A FREE BOUNDARY

The general problem with a free boundary is formulated as follows: at the initial instant of time we are given a region Ω and a velocity field $\mathbf{u}|_{t=0}=\mathbf{u}_0(\mathbf{x}), \operatorname{div}\mathbf{u}_0=0, \mathbf{x}\in\Omega$, and for $t > 0$ we try to find the region $\Omega(t)$ and determine the functions $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ which satisfy the Euler equations

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\nabla\mathbf{u}=-\nabla p+\mathbf{g},\quad \operatorname{div}\mathbf{u}=0 \tag{1.1}$$

in $\Omega(t)$ and satisfy the conditions

$$p|_{\Gamma(t)}=0,\quad \left.\frac{dF}{dt}\right|_{\Gamma(t)}=0$$

on the free boundary. Here $F(\mathbf{x}, t) = 0$ is the equation of the free boundary $\Gamma(t)$; $\mathbf{g}$ is the mass force vector.

We introduce the Lagrangian coordinates $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ by means of the equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{u}(\mathbf{x}, t),\quad \mathbf{x}|_{t=0}=\boldsymbol{\xi},\quad \boldsymbol{\xi}\in\Omega.$$

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If the function $\mathbf{x} = \mathbf{x}(\xi, t)$ is known, then the velocity vector can be found by simply differentiating with respect to t, and the pressure p can be found by integrating the first of the equations in (1.1). The region $\Omega(t)$ is found to be the image of the region Ω under the mapping $\xi \to \mathbf{x} = \mathbf{x}(\xi, t)$, and therefore, the problem of the motion of a liquid with a free boundary can be regarded as the problem of finding the mapping $\mathbf{x} = \mathbf{x}(\xi, t)$, $\xi \in \Omega$ in the fixed region Ω.

Ovsyannikov showed that the problem of finding the mapping $\mathbf{x} = \mathbf{x}(\xi, t)$, equivalent to the problem of the motion of a liquid with a free boundary, has in Lagrangian coordinates the form [5]

$$\Omega: \begin{cases} \operatorname{div} M^{-1}\mathbf{x}_t = 0; & (1.2) \\ M^*(\mathbf{x}_{tt} - \mathbf{g}) + \nabla p = 0; & (1.3) \end{cases}$$

$$\Gamma: M^*(\mathbf{x}_{tt} - \mathbf{g}) \cdot \tau = 0; \tag{1.4}$$

$$\left.\begin{array}{ll} \mathbf{x} = \xi, & (1.5) \\ \mathbf{x}_t = \mathbf{u}_0(\xi), \operatorname{div} \mathbf{u}_0 = 0 & (1.6) \end{array}\right\} t = 0.$$

Here M is the Jacobi matrix of the mapping $\xi \to \mathbf{x}$; M^{-1}, M^* are matrices which are, respectively, the inverse and the transpose of this matrix; τ is any displacement along the boundary Γ; all the operators are taken with respect to the Lagrangian coordinates.

2. EQUATIONS FOR SMALL PERTURBATIONS

Suppose that we know some solution of the problem (1.2)-(1.6) with the initial function $\mathbf{u}_0$, which we shall call the fundamental solution. Consider another solution in the same region Ω but with a changed initial function

$$\tilde{\mathbf{x}}_t(\xi) = \mathbf{u}_0(\xi) + \mathbf{v}(\xi). \tag{2.1}$$

This solution is called the perturbed solution, and the function $\mathbf{v}$ is called the initial perturbation. If the perturbed solution is described by the function $\tilde{\mathbf{x}}$, then the difference $\tilde{\mathbf{x}} - \mathbf{x}$ can be called the perturbation of the fundamental solution $\mathbf{x}$. Assuming that the initial disturbance $\mathbf{v}$ is small, we can hope that the perturbations will be small for at least a bounded interval of time, and we can proceed to study the evolution of small perturbations in the linear approximation.

We set $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{X}$ and denote by N the difference $\tilde{M} - M$ of the Jacobi matrices corresponding to the perturbed and fundamental solutions; then

$$\tilde{M} = M + N, \quad N = \frac{\partial \mathbf{X}}{\partial \xi}.$$

We consider Eq. (1.2) for the perturbed solution

$$\operatorname{div} \tilde{M}^{-1} \tilde{\mathbf{x}}_t = 0.$$

According to [5], in the linear approximation we obtain

$$\operatorname{div} M^{-1} \mathbf{X} = 0. \tag{2.2}$$

We now turn to the transformed equation (1.3). For the perturbed solution we have

$$\tilde{M}^*(\tilde{\mathbf{x}}_{tt} - \tilde{\mathbf{g}}) + \nabla \tilde{p} = 0,$$

where

$$\tilde{p} = p + M^{*-1} \nabla \mathbf{g} \cdot \mathbf{X} + P, \ \tilde{\mathbf{g}} = \mathbf{g}(\mathbf{x} + \mathbf{X}, t).$$

Making use of the identity $\nabla(\mathbf{g} \cdot \mathbf{X}) = \nabla(\mathbf{g})\mathbf{X} + N^* \mathbf{g}$,

$$\widetilde{M}^* (\widetilde{\mathbf{x}}_{tt} - \widetilde{\mathbf{g}}) + \nabla \widetilde{p} = M^* (\mathbf{x}_{tt} - \mathbf{g}) + \nabla p + M^* \mathbf{X}_{tt} - M_{tt}^* \mathbf{X} + \nabla [(\mathbf{x}_{tt} - \mathbf{g} + M^{*-1} \nabla p) \cdot \mathbf{X}] - A\mathbf{X} + \nabla P,$$

$$A = M^* \frac{\partial (\mathbf{g})}{\partial (\mathbf{x})} - \left(\frac{\partial (\mathbf{g})}{\partial (\xi)}\right)^*$$

in the linear approximation. Taking account of the fact that in the fundamental motion (1.3) is satisfied, we arrive at the relation

$$M^* \mathbf{X}_{tt} - [M_{tt}^* + A] \mathbf{X} = -\nabla P. \tag{2.3}$$

On the free boundary, condition (1.4) reduces to the following:

$$\nabla (P + M^{*-1} \nabla p \cdot \mathbf{X})|\cdot {}_{\Gamma}\tau = 0,$$

i.e., on the boundary Γ the quantity $P + M^{*-1} \Delta p \cdot X$ has a constant value, which may be a function of time t. Hence

$$(P + M^{*-1} \nabla p \cdot \mathbf{X})|_{\Gamma} = 0. \tag{2.4}$$

Equation (2.4) also follows from the conditions $\tilde{p}|\Gamma = 0$, $p|\Gamma = 0$.

To these equations we must add the initial conditions for t = 0, which follow from the fact that both solutions, the fundamental solution and the perturbed solution, satisfy the same initial condition (1.5) and also satisfy the condition (2.1), as a result of which

$$\mathbf{X}|_{t=0} = 0, \ \mathbf{X}_t|_{t=0} = \mathbf{v}, \ \operatorname{div} \mathbf{v} = 0. \tag{2.5}$$

Equations (2.2)-(2.4) describe the evolution of small perturbations with the initial conditions (2.5).

For our further transformations, we assume that A = 0 and we introduce a new function Φ by means of the equation

$$\Phi_t = -P.$$

It can be seen that the condition A = 0 is necessary and sufficient to make the vector of mass forces a potential vector in the space of the point (x) : $g = \nabla x^h$.

For A = 0, Eq. (2.3) has the general solution

$$\mathbf{X} = V \int_0^t V^{-1} M^{*-1} (\nabla \Phi + \mathbf{v})\, dt, \tag{2.6}$$

where V is the fundamental matrix of the corresponding homogeneous equation

$$V_t = M^{*-1} M_t^* V, \quad V|_{t=0} = E. \tag{2.7}$$

From Eq. (1.4) it follows that

$$M_t^* = M^* \left(\frac{\partial (\mathbf{u})}{\partial (\mathbf{x})}\right)^*,$$

and therefore we can simplify the equation for V and give it the form

$$V_t = \left(\frac{\partial (\mathbf{u})}{\partial (\mathbf{x})}\right)^* V, \quad V|_{t=0} = E. \tag{2.8}$$

From (2.8) it follows that if the fundamental motion is of potential type, the matrix V will coincide with the matrix M (M ≡ V), and the identity for such motions that was noted in [5] follows from (2.7).

Since the condition p = 0 is satisfied on the boundary (the boundary is free), the gradient of the pressure is directed along the normal to Γ, i.e.,

$$\nabla p|_\Gamma = \frac{\partial p}{\partial n}\Big|_\Gamma \mathbf{n}.$$

Substituting the vector **X** into (2.2), (2.4) and making use of the resulting equations, we arrive at the problem of the evolution of small perturbations of an arbitrary fundamental motion of a liquid with a free boundary, which takes place under the action of a small perturbation of the initial function (2.1):

$$\Omega: \operatorname{div}\left[M^{-1}V\int_0^t V^{-1}M^{*-1}(\nabla\Phi+\mathbf{v})\,dt\right]=0; \tag{2.9}$$

$$\Gamma: \Phi_t = \frac{\partial p}{\partial n}\mathbf{n}\cdot M^{-1}V\int_0^t V^{-1}M^{*-1}(\nabla\Phi+\mathbf{v})\,dt; \tag{2.10}$$

$$t=0: \Phi=0. \tag{2.11}$$

We assume that $\partial p/\partial n \neq 0$, $(\xi \in \Gamma, t \in [0, T]$, then from (2.9), (2.10) we obtain the equation

$$\int_\Gamma \left(\frac{\partial p}{\partial n}\right)^{-1}\Phi_t\, d\Gamma = 0, \tag{2.12}$$

which is valid for any solution of the problem (2.9)-(2.11).

In what follows we shall assume everywhere that $\partial p/\partial n \le -d < 0$ with constant $d > 0$. For potential motions of a liquid in the absence of mass forces this inequality is always valid [5] and means physically that the acceleration of the particles on the boundary Γ(t) is directed toward the interior of the liquid.

If we assume that the mass forces are of potential type, then there exists a Weber integral of the Euler equations [11]

$$\Omega: \begin{cases} M^*\mathbf{x}_t = \nabla\varphi + \mathbf{u}_0, \\ \operatorname{div} M^{-1}M^{*-1}(\nabla\varphi + \mathbf{u}_0) = 0; \end{cases}$$

$$\Gamma: \varphi_t = \frac{1}{2}|M^{*-1}(\nabla\varphi+\mathbf{u}_0)|^2 + h;$$

$$t=0: \begin{cases} \mathbf{x}=\xi, \varphi=0, \\ \mathbf{x}_t=\mathbf{u}_0, \operatorname{div}\mathbf{u}_0=0. \end{cases}$$

If for the perturbed solution we set

$$\tilde\varphi = \varphi + \mathbf{x}_t\cdot\mathbf{X} + \Phi, \quad \tilde{\mathbf{x}} = \mathbf{x}+\mathbf{X},$$

we can show that the function Φ satisfies (2.9)-(2.11). If we do not assume that the vector **g** is a potential vector, then we must make use of Eqs. (2.2)-(2.5).

3. EXISTENCE AND UNIQUENESS THEOREM

Suppose that the fundamental solution is defined in the region Ω with a boundary of class C^2 and belongs to the class C^3 in the cylinder $D = [0, T] \times \bar\Omega$.

Differentiating Eqs. (2.9), (2.10) once with respect to time, we obtain

$$-\operatorname{div} M^{-1}M^{*-1}\nabla\Phi = \operatorname{div}\left(\mathbf{q} - B\int_0^t V^{-1}M^{*-1}\nabla\Phi dt\right) \equiv f(\Phi),\quad \xi\in\Omega; \tag{3.1}$$

$$(a\Phi_t)_t + \mathbf{n}\cdot M^{-1}M^{*-1}\nabla\Phi = -\mathbf{n}\cdot\left(\mathbf{q} + B\int_0^t V^{-1}M^{*-1}\nabla\Phi dt\right) \equiv h(\Phi), \quad \xi\in\Gamma, \tag{3.2}$$

where we use the notation

$$\mathbf{q} = \left[M^{-1}M^{*-1} + B\int_0^t V^{-1}M^{*-1}dt\right]\mathbf{v},\ B = (M^{-1}V),$$

$$a = \left(-\frac{\partial p}{\partial n}\right)^{-1} \geqslant a_0 > 0.$$

Suppose that $\|,\ \|_{+,\Gamma},\ \|,\|_{0,\Gamma},\ \|,\|_{0,\Omega},\ \|,\|_{1,\Omega}$ are, respectively, the norms in the spaces $W_2^{1/2}(\Gamma)$, $L^2(\Gamma)$, $L^2(\Omega)$, $W_2^1(\Omega)$; l is an arbitrary element of the space $W_2^{1/2}(\Gamma)$.

LEMMA. For any $l\in W_2^{1/2}(\Gamma)$ there exists a unique solution of the problem

$$\begin{cases} -\operatorname{div} M^{-1}M^{*-1}\nabla\Phi = f(\Phi),\ \xi\in\Omega, \\ \Phi|_\Gamma = l,\ \xi\in\Gamma, \end{cases} \tag{3.3}$$

for which

$$\Phi\in L^2([0,T];\ W_2^1(\Omega)).$$

(The definition of the spaces $L^p([0, T]; H)$, $0 < p \le \infty$ and of derivatives with respect to t in these spaces is given in [10].)

The proof of the lemma will be given after the proof of the existence theorem. If the lemma is true, then according to [10] we can define the derivative along the conormal as an element of the space $W_2^{-1/2}(\Gamma)$: $\partial/\partial\nu = \mathbf{n}\cdot M^{-1}M^{*-1}\nabla$. We define the operator K, which satisfies with the function $\psi\in W_2^{1/2}(\Gamma)$ an element $K\psi\in W_2^{-1/2}(\Gamma)$ according to the following rule: for the function $\psi\in W_2^{1/2}(\Gamma)$ we find the solution of the problem (3.3) and then calculate $K\psi = \mathbf{n}\cdot M^{1-}M^{*-1}\nabla\Phi$. From the definition it follows that K acts from $W_2^{1/2}(\Gamma)$ into $W_2^{1/2}(\Gamma)$.

We set $\Phi(t)|_\Gamma = \psi(t)$; then

$$\frac{\partial\Phi(t)}{\partial\nu} = K(t)\psi(t),$$

and, consequently, the problem (3.1), (3.2) reduces to the Cauchy problem for an integrodifferential equation with an unbounded operator in Hilbert space for the function ψ on the free boundary Γ, namely,

$$\frac{d}{dt}\left(a\frac{d\psi}{dt}\right) + K(t)\psi = h(\psi); \tag{3.4}$$

$$\psi = \frac{d\psi}{dt} = 0 \quad \text{for} \quad t = 0. \tag{3.5}$$

To find the function Φ, all we need now to do is solve the Dirichlet problem

$$-\operatorname{div} M^{-1}M^{*-1}\nabla\Phi = f(\Phi),\ \xi\in\Omega;$$
$$\Phi(\xi,t) = \psi(\xi,t),\ \xi\in\Gamma.$$

THEOREM. If $\mathbf{v}\in L^2(\Omega)$, then there exists a solution, and, in fact, a unique solution, of the problem (3.4), (3.5) when

$$\begin{cases} \psi\in L^2([0,T];\ W_2^{1/2}(\Gamma)), \\ \psi'\in L^2([0,T];\ L^2(\Gamma)), \end{cases} \tag{3.6}$$

where the prime denotes differentiation with respect to t.

We use the Galerkin method to prove this. The fundamental feature of the proof is the derivation of the a priori estimate

$$\|\psi(t)\|^2_{+,\Gamma}+\|\psi'(t)\|^2_{0,\Gamma}\leqslant \text{const}, \tag{3.7}$$

in which the constant depends only on the region Ω and the fundamental solution.

In order to derive the a priori estimate (3.7), we assume that in addition to the inclusion (3.6), another condition is also satisfied: the function $(a\psi')'$ is square summable with respect to t in [0, T].

We multiply Eq. (3.4) by $\psi'(t)$ and integrate along Γ:

$$((a\psi')',\psi')_\Gamma+(K\psi,\psi')_\Gamma=(h(\psi),\psi')_\Gamma, \tag{3.8}$$

where

$$(\varphi,\psi)_\Gamma=\int\limits_\Gamma\varphi\psi d\Gamma.$$

The first term can be transformed as follows:

$$((a\psi')',\psi')_\Gamma=\frac{1}{2}\frac{d}{dt}(a\psi',\psi')_\Gamma+\frac{1}{2}(a'\psi',\psi')_\Gamma,$$

and to transform the second term we use the Gauss–Ostrogradsky formula

$$(K\psi,\psi')_\Gamma=-(\Phi_t,f(\Phi))_\Omega+\frac{1}{2}\frac{d}{dt}(M^{*-1}\nabla\Phi,M^{*-1}\nabla\Phi)_\Omega-\frac{1}{2}(\nabla\Phi,(M^{-1}M^{*-1})'\nabla\Phi)_\Omega.$$

Substituting these equations into (3.8) and integrating with respect to time, after some transformations we arrive at the relation

$$\begin{gathered}(a\psi',\psi')_\Gamma+(M^{*-1}\nabla\Phi,M^{*-1}\nabla\Phi)_\Omega=\\=\int\limits_0^t(\nabla\Phi,(M^{-1}M^{*-1})'\nabla\Phi)_\Omega dt-\int\limits_0^t(a'\psi',\psi')_\Gamma dt-2(\nabla\Phi,\mathbf{q})_\Omega-\\-2\left(\nabla\Phi,B\int\limits_0^t V^{-1}M^{*-1}\nabla\Phi dt\right)_\Omega+2\int\limits_0^t(\nabla\Phi,\mathbf{q}')_\Omega dt+\\+2\int\limits_0^t(\nabla\Phi,BV^{-1}M^{*-1}\nabla\Phi)_\Omega dt+2\int\limits_0^t\left(\nabla\Phi,B'\int\limits_0^\tau V^{-1}M^{*-1}\nabla\Phi d\tau\right)_\Omega dt.\end{gathered} \tag{3.9}$$

Let $\alpha=\min\limits_{\xi\in\Gamma,t\in[0,T]}a(\xi,t)>0$, then, obviously, $(a\psi',\psi')_\Gamma\geq\alpha\|\psi'\|^2_{0,\Gamma}$ and $(M^{*-1}\nabla\Phi,\ M^{*-1}\nabla\Phi)_\Omega\geqslant\beta\|\nabla\Phi\|^2_{0,\Omega}$, since for any vector τ, $|\tau|\neq0$, we have $(M^{*-1}\tau,M^{*-1}\tau)\geq\beta|\tau|^2$, and the left side of Eq. (3.9) has a lower bound:

$$(a\psi',\psi')_\Gamma+(M^{*-1}\nabla\Phi,M^{*-1}\nabla\Phi)_\Omega\geqslant\alpha\|\psi'\|^2_{0,\Gamma}+\beta\|\nabla\Phi\|^2_{0,\Omega}.$$

We estimate the right side by using the trace theorem [10], as follows:

$$\left|\int\limits_0^t(\nabla\Phi,(M^{-1}M^{*-1})'\nabla\Phi)_\Omega dt\right|\leqslant c_0\int\limits_0^t\|\Phi\|^2_{1,\Omega}dt\leqslant c_1\int\limits_0^t\|\psi\|^2_{+,\Gamma}dt,$$

$$\left|-\int\limits_0^t(a'\psi',\psi')_\Gamma dt\right|\leqslant c_2\int\limits_0^t\|\psi'\|^2_{0,\Gamma}dt,$$

$$|-2(\nabla\Phi,\mathbf{q})_\Omega|\leqslant\varepsilon_1\|\nabla\Phi\|^2_{0,\Omega}+\frac{c_3}{\varepsilon_1}\|\mathbf{v}\|^2_{0,\Omega},$$

$$\left|2\int_0^t (\nabla \Phi, \mathbf{q}')_\Omega\, dt\right| \leqslant c_4 \int_0^t \|\psi\|^2_{+,\Gamma}\, dt + c_5 \|\mathbf{v}\|^2_{0,\Omega},$$

$$\left|2\left(\nabla \Phi, B\int_0^t V^{-1}M^{*-1}\nabla \Phi dt\right)_\Omega\right| \leqslant c_6\varepsilon_2 \|\nabla \Phi\|^2_{0,\Omega} + c_7 \int_0^t \|\psi\|^2_{+,\Gamma}\, dt,$$

$$\left|2\int_0^t (\nabla \Phi, BV^{-1}M^{*-1}\nabla \Phi)_\Omega\, dt\right| \leqslant c_8 \int_0^t \|\psi\|^2_{+,\Gamma}\, dt,$$

$$\left|2\int_0^t \left(\nabla \Phi, B'\int_0^\tau V^{-1}M^{*-1}\nabla \Phi d\tau\right)_\Omega dt\right| \leqslant c_9 \int_0^t \|\psi\|^2_{+,\Gamma}\, dt,$$

where the $c_i > 0$ denote different constants. Summing the resulting estimates and selecting ε_1, ε_2 in such a way that $\gamma = \beta - (\varepsilon_1 + c_6\varepsilon_2) > 0$ (it is sufficient to take $\varepsilon_2 = \varepsilon_1/c_6$, $\varepsilon_1 < \beta/2$), we obtain the inequality

$$\gamma \|\nabla \Phi\|^2_{0,\Omega} + \alpha \|\psi'\|^2_{0,\Gamma} \leqslant m + c_{10}\int_0^t \left(\|\psi\|^2_{+,\Gamma} + \|\psi'\|^2_{0,\Gamma}\right) dt.$$

We add to each side of the last inequality $\sigma\|\psi\|^2_{0,\Gamma}$ with arbitrary $\sigma > 0$ and, noting that

$$\|\psi\|_{0,\Gamma} \leqslant \int_0^t \|\psi'(t)\|_{0,\Gamma}\, dt,$$

$$\gamma \|\nabla \Phi\|^2_{0,\Omega} + \sigma \|\psi\|^2_{0,\Gamma} \geqslant \lambda \|\Phi\|^2_{1,\Omega} \geqslant \omega \|\psi\|^2_{+,\Gamma}$$

with positive constants λ, ω, we finally obtain the inequality

$$\delta\left(\|\psi'(t)\|^2_{0,\Gamma} + \|\psi(t)\|^2_{+,\Gamma}\right) \leqslant m + c_{11}\int_0^t \left(\|\psi'(t)\|^2_{0,\Gamma} + \|\psi(t)\|^2_{+,\Gamma}\right) dt,$$

where $\delta = \min(\alpha, \omega)$. By using Gronwall's lemma, we obtain from this the estimate of (3.7) with a constant equal to

$$\frac{mT}{\delta}\exp\left(\frac{c_{11}T}{\delta}\right).$$

This proves the a priori estimate.

We define an approximate solution ψ_n of the problem (3.4), (3.5) by means of the equations

$$\left.\begin{aligned}
&\psi_n(t) = \sum_{i=1}^n c_{in}(t) w_i, \quad n = 1, 2, \ldots \\
&\left((a\psi_n')', w_j\right)_\Gamma + (K(t)\psi_n, w_j)_\Gamma = (h(\psi_n), w_j), \\
&\psi_n(0) = \psi'_n(0) = 0,
\end{aligned}\right\}$$

where w_j, $j = 1, 2, \ldots$, is a basis in $W_2^{1/2}(\Gamma)$. Then the unknown coefficients $c_{in}(t)$ are determined from a system of integrodifferential equations of second order. Therefore, the assumption made concerning the summability of $(a\psi_n')'$ is satisfied for the approximations, and the estimate of (3.7) holds for them. Then we can prove by a known method that the limit function is a solution of the problem (3.4), (3.5) and is unique [11].

To prove the lemma, we again use Galerkin's method. We continue l to Ω in such a way that the continued function $\tilde{l}$ will belong to $W_2^1(\Omega)$. We introduce the new function $\Psi = \Phi - \tilde{l}$; then $\Psi_\Gamma = 0$, and in the region Ω we obtain for Ψ the problem

$$-\operatorname{div} M^{-1}M^{*-1}\nabla \Psi = \operatorname{div}\left(\mathbf{q} + \mathbf{p} + B\int_0^t V^{-1}M^{*-1}\nabla \Psi dt\right), \Psi|_\Gamma = 0, \tag{3.10}$$

where

$$\mathbf{p}=M^{-1}M^{*-1}\nabla\tilde{l}+B\int_0^t V^{-1}M^{*-1}\nabla\tilde{l}\,dt.$$

We multiply Eq. (3.10) by Ψ and integrate over Ω:

$$(M^{*-1}\nabla\Psi,\ M^{*-1}\nabla\Psi)_\Omega=\left(\Psi,\ \operatorname{div}\left(\mathbf{b}+B\int_0^t V^{-1}M^{*-1}\nabla\Psi dt\right)\right)_\Omega,\quad \mathbf{b}=\mathbf{p}+\mathbf{q}. \tag{3.11}$$

Obviously,

$$(M^{*-1}\nabla\Psi,\ M^{*-1}\nabla\Psi)_\Omega\geqslant\beta\|\nabla\Psi\|_{0,\Omega}^2\geqslant c_1\|\Psi\|_{1,\Omega},$$

and the left side can be estimated as follows:

$$\left(\Psi,\ \operatorname{div}\left(\mathbf{b}+B\int_0^t V^{-1}M^{*-1}\nabla\Psi dt\right)\right)_\Omega\leqslant\left(\frac{1}{2}\varepsilon_1+\frac{c_2\varepsilon_2}{2}\right)\|\Psi\|_{1,\Omega}^2+\|\mathbf{b}\|_{0,\Omega}^2+c_3\int_0^t\|\Psi\|_{1,\Omega}^2\,dt.$$

Setting $\varepsilon_2=\varepsilon_1/c_2$, $\varepsilon_1=c_1/2$, we obtain the inequality

$$\frac{c_1}{2}\|\Psi\|_{1,\Omega}^2\leqslant n+c_3\int_0^t\|\Psi\|_{1,\Omega}^2\,dt,$$

and from this it follows that $\|\Psi(t)\|_{1,\Omega}^2\leq$ const when $0\leq t\leq T$.

Remarks. 1. From Eq. (3.4) it follows that

$$\frac{d^2\psi}{dt^2}\in L^2\left([0,T];\ W_2^{-1/2}(\Gamma)\right),$$

from this ([10], Theorem 3.1) it follows that ψ' is an almost-everywhere continuous function $[0,T]\to W_2^{-1/2}(\Gamma)$; then it follows from (3.6) that ψ is a continuous function $[0,T]\to L^2(\Gamma)$. Therefore, the equations in (3.6) are meaningful.

2. From the inequality (3.7) we can derive the stronger assertion

$$\psi\in L^\infty\left([0,T];\ W_2^{1/2}(\Gamma)\right);\quad \psi'\in L^\infty([0,T);\ L^2(\Gamma)).$$

3. The existence and uniqueness theorem for the case in which the fundamental motion is potential motion was proved by the author in [9].

4. STABILITY

At the present time we do not have any kind of general approach to the study of the behavior of perturbation as $t\to\infty$ of the problem (2.9)-(2.11), i.e., to the question of the stability of the fundamental solution with respect to changes in the initial data. Therefore, every problem of this kind must be considered individually. It should be noted that under the conditions of the stability problem, the estimate (3.7) becomes weaker as t increases and yields no result as $t\to\infty$, since the constant in (3.7) becomes infinite. The stability can be most precisely characterized by the behavior as $t\to\infty$ of the normal component of the perturbation vector

$$R=\mathbf{X}\cdot\mathbf{n}_x,\ \mathbf{n}_x \text{ — normal to } \Gamma(t),$$

i.e., by the deviation of the free boundary from its unperturbed state. By Eq. (2.4), we have

$$R=\mathbf{X}\cdot\mathbf{n}_x=\mathbf{X}\cdot\frac{\nabla_x F}{|\nabla_x F|}=M^{-1}\mathbf{X}\cdot\mathbf{n}\frac{|\nabla_\xi F|}{|M^{*-1}\nabla_\xi F|}=\left(\frac{\partial p}{\partial n}\right)^{-1}\frac{|\nabla_\xi F|}{|M^{*-1}\nabla_\xi F|}\,\Phi_t|_\Gamma. \tag{4.1}$$

Now we assume that the fundamental motion is a potential motion. As has already been remarked, in this case the matrix V coincides with the matrix M and the problem (2.9)-(2.11) becomes somewhat simpler. We have

$$\Omega:\operatorname{div} M^{-1}M^{*-1}(\nabla\Phi+\mathbf{v})=0; \tag{4.2}$$

$$\Gamma:\Phi_{tt}-\frac{\partial p_t}{\partial n}\Phi_t-\frac{\partial p}{\partial n}\mathbf{n}\cdot M^{-1}M^{*-1}(\nabla\Phi+\mathbf{v})=0; \tag{4.3}$$

$$t=0:\ \Phi=\Phi_t=0. \tag{4.4}$$

Here we consider the problem (4.2)-(4.4) in two special cases.

Example 1 (Stretching of a Strip). The mapping $\xi\to\mathbf{x}$, corresponding to the fundamental solution, will be taken in the form

$$x=\frac{\xi}{1-kt},\quad y=\eta(1-kt),\quad k=\text{const}. \tag{4.5}$$

The components of the velocities and the pressure are given by the formulas

$$u=\frac{k\xi}{(1-kt)^2}=\frac{kx}{1-kt};\quad v=-k\eta=-\frac{ky}{1-kt};$$
$$p=\frac{k^2}{(1-kt)^4}(\xi_0^2-\xi^2).$$

The interpretation of this solution is the following: the region Ω has the shape of a rectangle (the "block") $|\xi|\le\xi_0$, on which for t = 0 we are given the velocity field $u_0=k\xi=kx$, $v_0=-k\eta=-ky$. At time t = 0 the line $\eta=\pi$ (the "punch") suddenly begins to move in a translational fashion with velocity $w=-k\pi$, and the line $\eta=0$ (the "base") remains fixed. The lines $|\xi|=\xi_0$ form a free boundary, and the lines $\eta=0$, $\eta=\pi$ form impenetrable walls.

Substituting the mappings (4.5) into Eqs. (4.2)-(4.4), we obtain ($\mathbf{v}$ = u, v)

$$\alpha^4\Phi_{\xi\xi}+\Phi_{\eta\eta}+\alpha^4u_\xi+v_\eta=0;\quad |\xi|<\xi_0; \tag{4.6}$$

$$(\alpha^4\Phi_\alpha)_\alpha+2\xi\alpha^2(\Phi_\xi+u)=0,\ |\xi|=\xi_0; \tag{4.7}$$

$$\Phi=\Phi_\alpha=0,\ (\alpha=1-kt),\ \alpha=1. \tag{4.8}$$

From (4.1) for the normal component we obtain

$$R=X=\frac{\alpha^3}{2\xi_0}\Phi_\alpha,\quad Y=\alpha\int_1^\alpha\frac{1}{\alpha^2}(\Phi_\eta+v)\,d\alpha,\quad \mathbf{X}=(X,Y).$$

The condition that the walls are impenetrable reduces to the condition that Y = 0 when $\eta=0$, $\eta=\pi$, or

$$\Phi_\eta+v=0,\ \eta=0,\ \pi.$$

(The case when curl v = 0 was investigated in [4, 5]). The rest of the discussion is the same as in [5], and therefore it will not be given here. We note merely that the equation for the amplitude of R is inhomogeneous. For solutions which are even with respect to ξ, when the block is stretched, Am R ~ t^2, and for odd solutions Am R ~ t, which means instability. When the block is compressed to a line, the motion is stable.

If the perturbations are of potential type, then the motion is stable for solutions which are even with respect to ξ.

Example 2. This problem, in the case of potential perturbations which take account of surface-tension forces (and also without such forces), was considered in [7]. We take the fundamental solution in the form

$$\mathbf{x}=\left(\frac{1}{\sqrt{\gamma}}\xi,\ \frac{1}{\sqrt{\gamma}}\eta,\ \gamma\zeta\right),\quad p=-\frac{3\varkappa^2}{8\gamma^3}(\xi^2+\eta^2-\rho_0^2),\quad \gamma=1+\varkappa t,\ \varkappa=\text{const}. \tag{4.9}$$

The free boundary is the lateral surface of a cylinder. The cylinder is bounded by two impenetrable walls: the fixed wall $\zeta = 0$ and the movable wall $\zeta = h$. As t increases, when $\varkappa > 0$, the cylinder contracts to the axis $x = y = 0$, and when $\varkappa < 0$, it "swells" to infinity in a time $t^* = -1/\varkappa$. Substituting (4.9) into Eqs. (4.2)-(4.4), we obtain in the cylindrical coordinates $\xi = \rho\cos\theta$, $\eta = \rho\sin\theta$, $\xi = \xi$, $(\mathbf{v} = (u^\rho, u^\theta, w))$:

$$\Phi_{\rho\rho}+\frac{1}{\rho}\Phi_\rho+\frac{1}{\rho^2}\Phi_{\theta\theta}+\frac{1}{\gamma^3}\Phi_{\zeta\zeta}=\left(1-\frac{1}{\gamma^3}\right)w_\zeta,\quad \rho<\rho_0; \tag{4.10}$$

$$\frac{1}{\gamma}(\gamma^3\Phi_\gamma)_\gamma+\frac{3}{4}\rho\Phi_\rho=\frac{3}{4}\rho u^\theta,\quad \rho=\rho_0; \tag{4.11}$$

$$\Phi=\Phi_\gamma=0,\quad \gamma=1. \tag{4.12}$$

For the normal component of R we have

$$R=-\frac{4\gamma^{5/2}}{3\varkappa\rho_0}\Phi_\gamma|_{\rho=\rho_0}. \tag{4.13}$$

Setting

$$(\Phi,\ w_\zeta,\ u^\theta)=(A,f,g)\ \exp[i(n\zeta+\lambda\theta)],\ i=\sqrt{-1},$$

we obtain for the function $A(\rho, \gamma)$ an inhomogeneous equation on Γ:

$$\frac{1}{\gamma}(\gamma^3A_\gamma)_\gamma+\frac{3}{4}\rho_0\frac{I_\lambda'(\beta\rho_0)}{I_\lambda(\beta\rho_0)}A=\frac{3}{4}\rho_0\left[g+\left(1-\frac{1}{\gamma^3}\right)\int_0^{\rho_0}\frac{I_\lambda(\beta r)}{I_\lambda(\beta\rho_0)}f(r)\,dr\right], \tag{4.14}$$

$$I_\lambda'(\beta\rho_0)=\frac{d}{d\rho}I_\lambda'(\beta\rho)|_{\rho=\rho_0},\quad \beta=n\gamma^{-3/2}.$$

Here I_λ is the Bessel function. Investigation of Eq. (4.14) shows that as $\gamma \to \infty$, in the case of axisymmetric perturbations ($\lambda = 0$), the normal component of R increases beyond all bounds, whereas it is bounded for potential perturbations [7]. If $\gamma \to 0$, the motion is stable, i.e., such perturbations are damped out as we approach the walls $\zeta = 0$, $\zeta = h$.

The above two examples show the destabilizing influence of nonpotential perturbations on the stability of the motion.

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